Limite de funcții

333. $\lim\limits_{x\to 0} x\cdot e^{-\frac{1}{x}}$

$\lim\limits_{x\searrow 0} x\cdot e^{-\frac{1}{x}} = 0\cdot e^{-\infty} = 0$

$\lim\limits_{x\nearrow 0} x\cdot e^{-\frac{1}{x}} \overset{-\frac{1}{x}=t}{=} \lim\limits_{t\to\infty} -\frac{1}{t}\cdot e^{t} = -\infty.$

Obs $(f^g)' = g\cdot f^{g-1}\cdot f' + f^g\cdot g'\cdot \ln f$

334. $\lim\limits_{x\to\infty}\left(x\left(1+\frac{1}{x}\right)^x - e^x\right) \overset{\frac{1}{x}=t}{=} \lim\limits_{t\to 0}\left(\frac{1}{t}(1+t)^{\frac{1}{t}} - \frac{e}{t}\right) =$

$= \lim\limits_{t\to 0}\frac{(1+t)^{\frac{1}{t}} - e}{t} \overset{\frac{0}{0}}{=} \lim\limits_{t\to 0}\frac{\frac{1}{t}\cdot(1+t)^{\frac{1}{t}-1} + (1+t)^{\frac{1}{t}}\cdot\left(-\frac{1}{t^2}\right)\ln(1+t)}{1} =$

$= \lim\limits_{t\to 0}\underbrace{(1+t)^{\frac{1}{t}-1}}_{e}\cdot\left[\frac{1}{t} - \frac{1}{t^2}\cdot(1+t)\ln(1+t)\right] =$

$= e\lim\limits_{t\to 0}\frac{t-(1+t)\ln(1+t)}{t^2} \overset{\frac{0}{0}}{=} e\lim\limits_{t\to 0}\frac{1-\ln(1+t)-1}{2t} =$

$= -\frac{e}{2}\cdot\lim\limits_{t\to 0}\frac{\ln(1+t)}{t} = -\frac{e}{2}$

$\underline{\text{Obs.}}$ $\lim_{t\to 0} \frac{(1+t)^{\frac{1}{t}} - e}{t} = -\frac{e}{2}$

$\underline{339.}$ $\lim_{x\to 0} \left(\frac{(1+x)^{\frac{1}{x}}}{e} \right)^{\frac{1}{x}} = \lim_{x\to 0} \left(1 + \frac{(1+x)^{\frac{1}{x}} - e}{e} \right)^{\frac{1}{x}} =$

$= \lim_{x\to 0} \left(1 + \frac{(1+x)^{\frac{1}{x}} - e}{e} \right)^{\frac{e}{(1+x)^{\frac{1}{x}} - e} \cdot \frac{(1+x)^{\frac{1}{x}} - e}{ex}} =$

$= e^{\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{ex}} = e^{\frac{1}{e} \cdot -\frac{e}{2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

$\underline{341}$ $\lim_{x\to 0} \left(1 + \frac{\operatorname{tg} x - x}{x} \right)^{\frac{1}{\sin^2 x}} = \lim_{x\to 0} \left(1 + \frac{\operatorname{tg} x - x}{x} \right)^{\frac{x}{\operatorname{tg} x - x} \cdot \frac{\operatorname{tg} x - x}{x \cdot \sin^2 x}} =$

$= e^{\lim_{x\to 0} \frac{\operatorname{tg} x - x}{x \cdot \sin^2 x}} = e^{\lim_{x\to 0} \frac{\operatorname{tg} x - x}{x^3}} = e^{\lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{3x^2}} =$

$= e^{\lim_{x\to 0} \frac{1 - \cos^2 x}{3x^2 \cdot \underset{\downarrow}{\cos^2 x}_{1}}} = e^{\lim_{x\to 0} \frac{\sin^2 x}{3x^2}} = e^{\frac{1}{3}}$

342 $\lim_{x\to\infty}\left(x-\sqrt{x^2+x+1}\cdot\frac{\ln(e^x+x)}{x}\right)=$

$=\lim_{x\to\infty}\left(x-\ln(e^x+x)\right)+\lim_{x\to\infty}\left(\ln(e^x+x)-\sqrt{x^2+x+1}\cdot\frac{\ln(e^x+x)}{x}\right)=$

$=\lim_{x\to\infty}\ln\frac{e^x}{e^x+x}+\lim_{x\to\infty}\left(1-\frac{\sqrt{x^2+x+1}}{x}\right)\cdot\ln(e^x+x)=$

$=\ln 1+\lim_{x\to\infty}\frac{x-\sqrt{x^2+x+1}}{x}\cdot\ln(e^x+x)=$

$=\lim_{x\to\infty}\frac{-x-1}{\left(x+\sqrt{x^2+x+1}\right)}\cdot\frac{\ln(e^x+x)}{x}=$

$=\lim_{x\to\infty}\frac{x\left(-1-\frac{1}{x}\right)}{x\left(1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\right)}\cdot\frac{\ln(e^x+x)}{x}=-\frac{1}{2}\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}=$

L'H

$=-\frac{1}{2}\lim_{x\to\infty}\frac{e^x+1}{e^x+x}=-\frac{1}{2}\lim_{x\to\infty}\frac{\cancel{e^x}\left(1+\frac{1}{e^x}\right)}{\cancel{e^x}\left(1+\underbrace{\frac{x}{e^x}}_{0}\right)}=-\frac{1}{2}$

346

$$\lim_{n\to\infty}\left(\lim_{x\to 0}\left(1+\operatorname{tg}^2 x+\operatorname{tg}^2(2x)+\ldots+\operatorname{tg}^2(nx)\right)^{\frac{1}{n^3\cdot x^2}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{x\to 0}\left(1+\operatorname{tg}^2 x+\ldots+\operatorname{tg}^2(nx)\right)^{\frac{1}{\operatorname{tg}^2 x+\ldots+\operatorname{tg}^2(nx)}\cdot\frac{\operatorname{tg}^2 x+\ldots+\operatorname{tg}^2(nx)}{n^3x^2}}\right)=$$

$$=\lim_{n\to\infty}\left(e^{\lim_{x\to 0}\frac{1}{n^3}\left(\frac{\operatorname{tg}^2 x}{x^2}+\ldots+\frac{\operatorname{tg}^2(nx)}{x^2}\right)}\right)=$$

$$=\lim_{n\to\infty}\left(e^{\frac{1}{n^3}\lim_{x\to 0}\left(\frac{\operatorname{tg}^2 x}{x^2}+2^2\cdot\frac{\operatorname{tg}^2(2x)}{(2x)^2}+\ldots+n^2\cdot\frac{\operatorname{tg}^2(nx)}{(nx)^2}\right)}\right)=$$

$$=\lim_{n\to\infty} e^{\frac{1}{n^3}\cdot\left(1+2^2+\ldots+n^2\right)}=e^{\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}}=$$

$$=e^{\frac{1}{3}}$$

328. $$\lim_{x\to 0}\frac{\sum_{k=1}^{n}\operatorname{arctg} k^2\cdot x}{\sum_{k=1}^{n}\ln(1+k^3x)}=\lim_{x\to 0}\frac{\operatorname{arctg} x+\operatorname{arctg} 2^2\cdot x+\ldots+\operatorname{arctg} n^2x}{\ln(1+x)+\ln(1+2^3x)+\ldots+\ln(1+n^3x)}$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{1}{1+x^2}+\frac{2^2}{1+(2^2x)^2}+\ldots+\frac{n^2}{(1+n^2x)^2}}{\frac{1}{1+x}+\frac{2^3}{1+2^3x}+\ldots+\frac{n^3}{1+n^3x}}=\frac{1+2^2+\ldots+n^2}{1+2^3+\ldots+n^3}=$$

$$= \frac{n(n+1)(2n+1)}{6} \cdot \frac{4}{n^2(n+1)^2} = \frac{2(2n+1)}{3n(n+1)}$$

<u>329</u> $$\lim_{x\to 0} \frac{a_1^{x} \cdot a_2^{2x} \cdot \ldots \cdot a_n^{nx} - 1}{x} = \lim_{x\to 0} \frac{a_1^{x} \cdot a_2^{2x} \cdot \ldots \cdot a_n^{nx} - a_1^{x} + a_1^{x} - 1}{x} =$$

$$= \lim_{x\to 0} \frac{\overset{\to 1}{a_1^{x}} \cdot \left(a_2^{2x} \cdot \ldots \cdot a_n^{nx} - 1\right)}{x} + \lim_{x\to 0} \frac{a_1^{x} - 1}{x} =$$

$$= \lim_{x\to 0} \frac{a_2^{2x} \cdot \ldots \cdot a_n^{nx} - a_2^{2x} + a_2^{2x} - 1}{x} + \ln a_1 =$$

$$= \lim_{x\to 0} \frac{\overset{\to 1}{a_2^{2x}} \cdot \left(a_3^{3x} \cdot \ldots \cdot a_n^{nx} - 1\right)}{x} + 2 \lim_{x\to 0} \frac{a_2^{2x} - 1}{2x} + \ln a_1 = \ldots$$

$$= n \ln a_n + (n-1) \ln a_{n-1} + \ldots + 2 \ln a_2 + \ln a_1 =$$

$$= \ln a_n^{n} \cdot \ldots \cdot a_2^{2} \cdot a_1$$

355. $$\lim_{x\to\infty}\frac{(e^x+x)^{n+1}-e^{(n+1)x}}{xe^{nx}}=\lim_{x\to\infty}\frac{\cancel{e^{(n+1)x}}^{\,e^x}\cdot\left[\left(\frac{e^x+x}{e^x}\right)^{n+1}-1\right]}{\cancel{xe^{nx}}}=$$

$$=\lim_{x\to\infty}\frac{\left(\frac{e^x+x}{e^x}\right)^{n+1}-1}{\frac{x}{e^x}}\overset{\frac{0}{0}}{=\!=}$$

$$=\lim_{x\to\infty}\frac{(n+1)\cdot\overbrace{\left(\frac{e^x+x}{e^x}\right)}^{\to 1}{}^{n}\cdot\frac{(e^x+1)e^x-(e^x+x)e^x}{e^{2x}}}{\frac{e^x-xe^x}{e^{2x}}}=$$

$$=(n+1)\lim_{x\to\infty}\frac{e^x(1-x)}{e^x(1-x)}=n+1$$

448. $$\lim_{x\to\infty}\frac{x\cdot F(x)}{e^{x^2}}=\lim_{x\to\infty}\frac{F(x)}{\frac{e^{x^2}}{x}}\overset{\frac{\infty}{\infty}}{=\!=}$$

$$=\lim_{x\to\infty}\frac{F'(x)}{\frac{2x^2\cdot e^{x^2}-e^{x^2}}{x^2}}=\lim_{x\to\infty}\frac{e^{x^2}}{\frac{e^{x^2}(2x^2-1)}{x^2}}=\lim_{x\to\infty}\frac{x^2}{2x^2-1}=$$

$$=\frac{1}{2}$$

Obs! $\lim_{x\to\infty} F(x) = \infty$, deoarece:

$$x^2 > x \Rightarrow \int e^{x^2} dx > \int e^{x} dx \Rightarrow \lim_{x\to\infty} \int e^{x^2} dx > \lim_{x\to\infty} \int e^{x} dx$$

iar $\lim_{x\to\infty} \int e^{x} dx = \infty$.

345 $\lim_{n\to\infty} n^2 \cdot \left(e^{\frac{1}{n+1}} - e^{\frac{1}{n}}\right)$.

Fie $f(x) = e^{\frac{1}{x}}$ și aplicăm teorema lui Lagrange pe intervalul $[n, n+1]$: $\exists\ c \in (n, n+1)$ astfel încât

$$\left.\begin{array}{l} f(n+1) - f(n) = f'(c) \\ f'(x) = -\frac{1}{x^2} e^{\frac{1}{x}} \end{array}\right\} \Rightarrow f(n+1) - f(n) = -\frac{1}{c^2} \cdot e^{\frac{1}{c}} \quad (1)$$

Din (1) $\Rightarrow \lim_{n\to\infty} n^2 \cdot \left(-\frac{1}{c^2}\right) e^{\frac{1}{c}} = \lim_{n\to\infty} \frac{-n^2}{c^2} \cdot e^{\frac{1}{c}}$

Dar $n < c < n+1 \Rightarrow \frac{1}{n+1} < \frac{1}{c} < \frac{1}{n} \Rightarrow$

$$\Rightarrow \frac{1}{(n+1)^2} < \frac{1}{c^2} < \frac{1}{n^2} \Rightarrow -\frac{n^2}{n^2} < -\frac{n^2}{c^2} < -\frac{n^2}{(n+1)^2}$$

-1

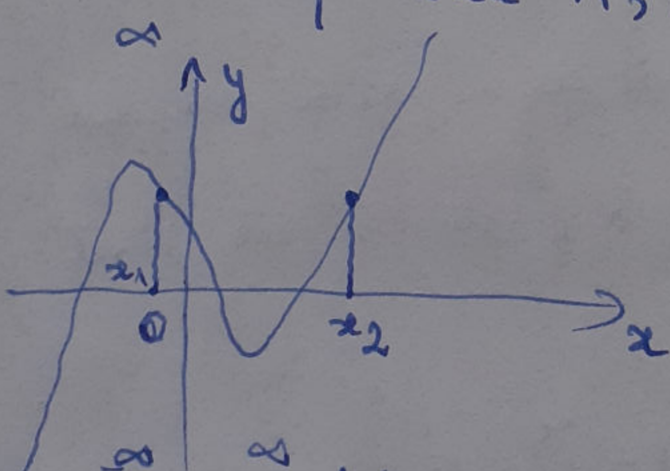
De asemenea $e^{\frac{1}{n+1}} < e^{\frac{1}{c}} < e^{\frac{1}{n}}$.

1

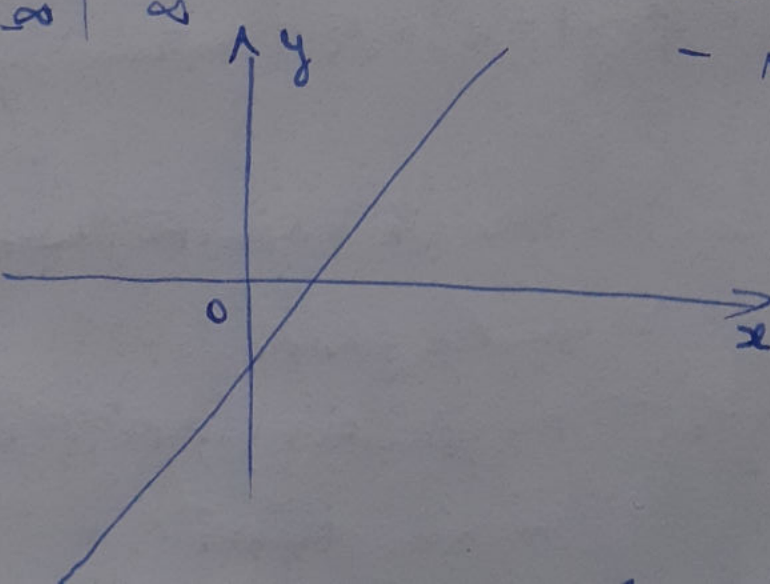
Prin urmare, $\lim\limits_{n\to\infty} -\frac{n^2}{c^2}\cdot e^{\frac{1}{c}} = -1\cdot 1 = -1$.

352. Dacă funcția $f:\mathbb{R}\to\mathbb{R}$ este continuă iar $\lim\limits_{x\to-\infty} f(x) = -\infty$ și $\lim\limits_{x\to\infty} f(x) = \infty$, singura concluzie care se poate trage, este că ea este surjectivă.

Mai jos, exemple în care ipotezele sunt îndeplinite dar răspunsurile A, B, D, E nu sunt adevărate.



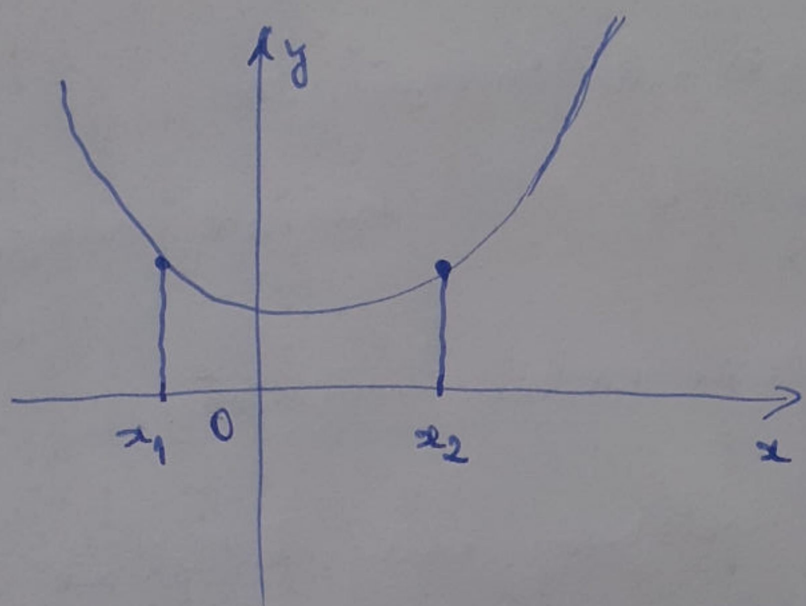
- nu este crescătoare
- nu este nici injectivă ($\exists\, x_1 \neq x_2$ a.î. $f(x_1) = f(x_2)$)
- nu este inversabilă



- poate fi injectivă

353. Dacă $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \infty$, singurul lucru pe care îl putem afirma este că funcția nu este injectivă.

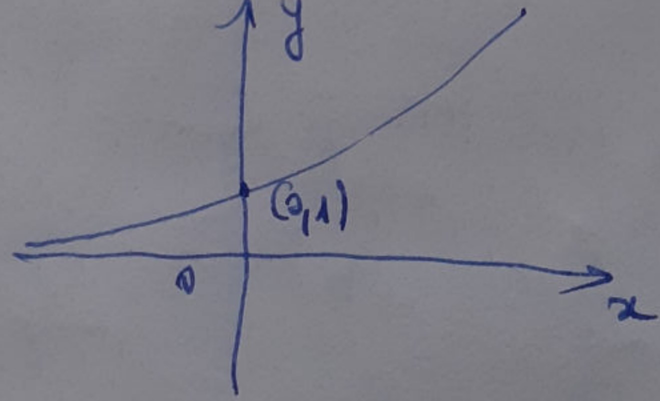
Mai jos, imagini care să elimine celelalte cazuri:



- nu este descrescătoare
- nu este surjectivă ($f: \mathbb{R} \to \mathbb{R}$!)
- nu este injectivă $\Rightarrow$ $\Rightarrow$ nu este inversabilă

354. Dacă $f: \mathbb{R} \to \mathbb{R}$, f continuă și injectivă, singurul lucru care se poate afirma despre ea, este că este strict monotonă.

De exemplu $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ este continuă și injectivă:



- nu este surjectivă ($\operatorname{Im} f = (0, \infty)$)
- nu are zerouri
- nu este inversabilă ($f: \mathbb{R} \to \mathbb{R}$!)
- nu este impară